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The decomposition of forms and cohomology of generalized complex manifolds

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Abstract

We study the decomposition of forms induced by a generalized complex structure giving a complete description of the bundles involved and, around regular points, of the operators ∂ and $\overline{\partial}$ associated to the generalized complex structure. We prove that if the generalized $\partial\overline{\partial}$-lemma holds then the decomposition of forms gives rise to a decomposition of the cohomology of the manifold, $H^\bullet(M) = \oplus_{-n}^{n} GH^k(M)$, and the canonical spectral sequence degenerates at E_1. We also show that if the generalized $\partial\overline{\partial}$-lemma holds, any generalized complex submanifold can be associated to a Poincaré dual cohomology class in the middle cohomology space $GH^0(M)$.

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1. Introduction

There aren't many simple ways to tell apart a complex or a symplectic manifold from an ordinary manifold. Except for some easy topological constraints, one has to use more advanced tools like Seiberg–Witten invariants to tackle the problem. For a compact Kähler manifold (M, ω, I), however, there is a handful of simple topological properties which can be used effectively. For example, the Strong Lefschetz Property says that the 2-form ω gives isomorphisms

$$[\omega]^k : H^{n-k}(M; \mathbb{R}) \stackrel{\cong}{\to} H^{n+k}(M; \mathbb{R}).$$

Another example is given by the $\partial\overline{\partial}$-lemma, which implies formality, evenness of the 'odd' Betti numbers b_{2k+1}, degeneracy of the Frölicher spectral sequence and decomposition of the complex cohomology into $H^{p,q}(M)$.

In this paper we investigate properties similar to the ones above for generalized complex manifolds, as introduced by Nigel Hitchin [8] and studied by Marco Gualtieri [6]. Recall that the concept of generalized complex

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structure unifies both complex and symplectic structures by searching for complex structures in $TM \oplus T^*M$ and many of the objects existing in complex geometry have their analogue in the generalized complex world. This includes Kähler manifolds, Calabi–Yau manifolds, submanifolds, the differential operators ∂ and $\overline{\partial}$ and the (p, q) decomposition of forms.

While the existence of a decomposition of differential forms into the ik-eigenspaces of the Lie algebra action of the generalized complex structure was established in Gualtieri's thesis, he provided no formula for the subbundles giving the decomposition. We establish a concrete description of the decomposition of forms in a generalized complex manifold, showing that the $\overline{\partial}$-cohomology of a generalized complex manifold is isomorphic to $\oplus_{p-q=k} H^p(M, \Omega^q)$ in the case of a complex manifold and to the ordinary cohomology with a shift in degree for a symplectic manifold. As an application of these expressions, we prove that the spectral sequence associated to the splitting $d = \partial + \overline{\partial}$ always degenerates at the first term in a symplectic manifold.

Still guided by the complex case, we study implications of the requirement that the generalized complex manifold satisfies the generalized $\partial\overline{\partial}$-lemma, i.e.,

$$\ker \partial \cap \operatorname{Im} \overline{\partial} = \ker \overline{\partial} \cap \operatorname{Im} \partial = \operatorname{Im} \partial\overline{\partial}.$$

This property should be of interest for some reasons. In the case of a generalized complex structure induced by a complex one, it is just the ordinary $\partial\overline{\partial}$-lemma, which carries so many topological implications and holds for Kähler manifolds. Recently, by finding a suitable generalization of Hodge theory for generalized Kähler manifolds [7], Gualtieri proved that generalized Kähler manifolds satisfy the generalized $\partial\overline{\partial}$-lemma with respect to both of the generalized complex structures involved. Another instance where this property manifests itself is in the symplectic setting, where Merkulov proved that it is equivalent to the Strong Lefschetz Property [11].

We show that, as in the original $\partial\overline{\partial}$-lemma for complex manifolds, if a generalized complex manifold has this property, then its cohomology decomposes according to the decomposition of forms into the ik-eigenspaces of the generalized complex structure. The generalized $\partial\overline{\partial}$-lemma also implies that the spectral sequence associated to the decomposition $d = \partial + \overline{\partial}$ degenerates at E_1. In contrast to the original $\partial\overline{\partial}$-lemma, this one does not seem to imply that b_{2k+1} is even and previous results from the author show it is not related to formality [2].

This paper is organized as follows. In the first section we introduce generalized complex manifolds and the decomposition of forms induced by a generalized complex structure as well as the decomposition of the exterior derivative $d = \partial + \overline{\partial}$. Then we study in detail these decompositions in the complex and symplectic cases as well as the effect of B-field transforms on them. Together with Gualtieri's version of Darboux's Theorem, this provides a complete local description of the subbundles of the exterior forms and, around regular points, of the operators ∂ and $\overline{\partial}$. In Section 4 we introduce the $\partial\overline{\partial}$-lemma and prove that if it holds, the decomposition of differential forms gives rise to a decomposition of the cohomology of the manifold. In Section 5 we introduce a spectral sequence for the splitting $d = \partial + \overline{\partial}$ and prove that this sequence degenerates at E_1 for example if the $\partial\overline{\partial}$-lemma holds or if the generalized complex structure is induced by a symplectic structure. We finish proving that, if the $\partial\overline{\partial}$-lemma holds, we can associate to each generalized complex submanifold a Poincaré dual cohomology class lying in the $GH^0(M)$ part of the decomposed cohomology.

2. Generalized complex geometry

The usual descriptions of complex structures have their analogue when defining generalized complex structures on a manifold M^{2n}, the main difference being that generalized complex structures put TM and T^*M in the same level.

Before giving the definition we need three vital ingredients. The first is the *natural pairing* in $TM \oplus T^*M$:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)).$$

To describe the next, we introduce a linear map σ on $\wedge^\bullet T^*M$ which acts on decomposables by

$$\sigma(e_1 \wedge \cdots \wedge e_j) = e_j \wedge \cdots \wedge e_1. \tag{2.1}$$

Definition. Given two forms of mixed degree $\varphi_i = \sum \varphi_i^k$, $i = 1, 2$, with $\deg(\varphi_i^k) = k$ in a n-dimensional vector space we define their *Mukai pairing*, (φ_1, φ_2) by

$$(\varphi_1, \varphi_2) = (\sigma(\varphi_1) \wedge \varphi_2)_{\mathrm{top}},$$

where top indicates the degree n component of the product.

Observe that the Clifford algebra of $TM \oplus T^*M$ with the natural pairing acts on forms via

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho,$$

and one can easily check that this action relates to the Mukai pairing by

$$((X + \xi) \cdot \varphi_1, (X + \xi) \cdot \varphi_2) = \xi(X)(\varphi_1, \varphi_2). \tag{2.2}$$

The Clifford action of $TM \oplus T^*M$ on forms also plays a role in the definition of the third ingredient.

Definition. The *Courant bracket* of $v_1, v_2 \in C^\infty(TM \oplus T^*M)$ is defined by the identity

$$2[v_1, v_2] \cdot \rho = d((v_1 v_2 - v_2 v_1) \cdot \rho) + (v_1 v_2 - v_2 v_1) \cdot d\rho + 2v_1 \cdot d(v_2 \cdot \rho) - 2v_2 \cdot d(v_1 \cdot \rho).$$

Spelling it out, we have

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\eta(X) - \xi(Y)).$$

Now we can define a generalized complex structure in a manifold M^{2n} in three equivalent ways.

Definition. A *generalized complex structure* is determined by any of the following three equivalent objects:

(i) An automorphism $\mathcal{J}$ of $TM \oplus T^*M$ which squares to -1 and is orthogonal with respect to the natural pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(Y));$$

and has vanishing Nijenhuis 'tensor', i.e., for all $v_1, v_2 \in C^\infty(TM \oplus T^*M)$

$$N(v_1, v_2) := -[\mathcal{J}v_1, \mathcal{J}v_2] + \mathcal{J}[\mathcal{J}v_1, v_2] + \mathcal{J}[v_1, \mathcal{J}v_2] + [v_1, v_2] = 0.$$

(ii) A subbundle $L < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ which is maximal isotropic with respect to the natural pairing, involutive with respect to the Courant bracket and satisfies $L \cap \overline{L} = \{0\}$;

(iii) A line subbundle of $\wedge^\bullet T^*_{\mathbb{C}}M$ generated at each point by a form of the form $\rho = \mathrm{e}^{B+i\omega} \wedge \Omega$, such that

$$(\rho, \overline{\rho}) = \Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0,$$

where B and ω are real 2-forms and Ω is a decomposable complex k-form and

$$d\rho = v \cdot \rho,$$

for some $v \in C^\infty(T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M)$.

Using the third description, the degree of Ω at a point is the *type* of the generalized complex structure at that point and the line bundle defining the generalized complex structure is the *canonical line bundle*. The points where the type is locally constant are called *regular points*.

Example 2.1. Let (M, I) be a complex manifold and define $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ by

$$\mathcal{J} = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}.$$

One can easily check that $\mathcal{J}^2 = -\mathrm{Id}$ and that $\mathcal{J}$ is orthogonal with respect to the natural pairing. The $+i$-eigenspace of $\mathcal{J}$ is $L = \wedge^{0,1} TM \oplus \wedge^{1,0} T^*M$, which is a maximal isotropic subspace of $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$.

The canonical bundle of this generalized complex structure is $\wedge^{n,0} T^*M$, the canonical bundle of the complex structure. If Ω is a local section of this bundle, then the integrability of the complex structure I is equivalent to

$$d\Omega = \xi \wedge \Omega,$$

for some (0,1)-form ξ. Therefore the induced generalized complex structure is also integrable.

Example 2.2. A symplectic structure ω on a manifold M also induces a generalized complex structure on M by letting

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The $+i$ eigenspace of $\mathcal{J}$ is given by

$$L = \{X - i\omega(X, \cdot) : X \in T_{\mathbb{C}}M\},$$

which has Clifford annihilator $e^{i\omega}$. This last expression shows clearly that the structure is integrable.

Example 2.3. A generalized complex structure $\mathcal{J}$ on a manifold can be deformed by a real closed 2-form B, a.k.a. *B-field*:

$$\mathcal{J}^B = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}.$$

If L is the $+i$-eigenspace of $\mathcal{J}$ then L^B, the $+i$-eigenspace of $\mathcal{J}^B$, is given by

$$\{X + \xi - B(X, \cdot) : X + \xi \in L\}.$$

Finally, if ρ is a local section of the canonical bundle of $\mathcal{J}$, then $e^B \wedge \rho$ is a local section of the canonical bundle of $\mathcal{J}^B$ and again the integrability condition is clear from the point of view of differential forms.

Products of generalized complex manifolds are still generalized complex with the obvious induced structure and by Example 2.3, B-field transforms of those are also generalized complex. Gualtieri proved that the previous examples actually provide a local model for a generalized complex structure around regular points:

Theorem 2.1 (*Gualtieri [6], Theorem 4.35*). *In a neighbourhood of any regular point there is a set of local coordinates such that the generalized complex structure is a B-field transform of the standard structure in $\mathbb{C}^{n-k} \times \mathbb{R}^{2k}$.*

The natural pairing gives an isomorphism $\overline{L} \cong L^*$ and hence $L \oplus L^* \cong L \oplus \overline{L} = T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$. So we have isomorphic Clifford algebras $\mathrm{Clif}(L \oplus L^*) \cong \mathrm{Clif}(T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M)$ and both $\wedge^\bullet L^*$ and $\wedge^\bullet T^*_{\mathbb{C}}M$ give concrete models for the space of spinors.

A generalized complex structure in a manifold M^{2n} induces a decomposition of the complex of differential forms in the following way. Let ρ be a local section of the canonical bundle and define a map of Clifford modules

$$\tau : \wedge^\bullet \overline{L} \to \wedge^\bullet T^*_{\mathbb{C}}M; \qquad \tau(1) = \rho.$$

One can easily check that τ is well defined (although it depends on the particular trivialization ρ of the canonical bundle) and the decomposition of $\wedge^\bullet \overline{L}$ by degree gives rise to a decomposition of forms in $\wedge^\bullet T^*_{\mathbb{C}}M$ by letting $U^k = \tau(\wedge^{n-k} \overline{L})$. So U^n is the canonical bundle and

$$U^{n-k} = \wedge^k \overline{L} \cdot U^n.$$

The space U^k can also be seen as the ik-eigenspace of the Lie algebra action of $\mathcal{J}$ (see [6]).

Observe that the choice of a trivialization of the canonical bundle gives rise to a map $\varphi : \wedge^{2n} \overline{L} \to \wedge^{2n} T^*_{\mathbb{C}}M$ in the following way. Let $\zeta \in \wedge^{2n} \overline{L}$ be such that $\tau(\zeta) = \overline{\rho}$ and then define $\varphi(\zeta) = (\overline{\rho}, \rho)$. With this definition and Eq. (2.2), one can easily check that for $\psi_i \in \wedge^\bullet \overline{L}$

$$\varphi((\psi_1, \psi_2)) = (\tau(\psi_1), \tau(\psi_2)).$$

In particular, as the Mukai pairing is trivial in $\wedge^i \overline{L} \times \wedge^j \overline{L}$, unless $i + j = 2n$, the Mukai pairing is also trivial in $U^k \times U^l$, unless $k = -l$. Therefore, we have the following:

Lemma 2.1. *The Mukai pairing vanishes in $U^k \times U^l$, unless, $k = -l$, in which case it is nondegenerate.*

Letting $\mathcal{U}^k = C^\infty(U^k)$, the integrability condition implies that

$$d : \mathcal{U}^k \to \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1},$$

which allows us to define operators ∂ and $\overline{\partial}$ via the projections

$$\partial : \mathcal{U}^k \to \mathcal{U}^{k+1} \qquad \overline{\partial} : \mathcal{U}^k \to \mathcal{U}^{k-1}.$$

See [6], Theorem 4.23, for a proof.

The real operator $d^{\mathcal{J}} = -i(\partial - \overline{\partial})$ will also be used in this paper. It can be equally defined as $d^{\mathcal{J}} = \mathcal{J}^{-1} d \mathcal{J}$, where $\mathcal{J}$ acts via the Lie group action, i.e., $\mathcal{J}\alpha = i^k \alpha$, for $\alpha \in U^k$ or as $d^{\mathcal{J}} = [d, \mathcal{J}] = d\mathcal{J} - \mathcal{J}d$, where now $\mathcal{J}$ acts via the Lie algebra action, i.e., for $\alpha \in U^k$, $\mathcal{J}\alpha = ik\alpha$.

2.1. The complex decomposition

A description of the spaces U^k and operators ∂ and $\overline{\partial}$ in the case of a generalized complex structure induced by a complex structure was presented by Gualtieri in his thesis [6].

According to Example 2.1, the canonical bundle of the generalized complex structure is just $\wedge^{n,0} T^* M$ and $\overline{L} = T^{1,0} M \oplus T^{*0,1} M$. Therefore one can easily check that

$$U^k = \oplus_{p-q=k} \wedge^{p,q} T^* M.$$

And the decomposition $d : \mathcal{U}^k \to \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1}$ furnishes the usual decomposition $d = \partial + \overline{\partial}$ in the complex manifold, therefore justifying the notation.

In this case, the $\overline{\partial}$-cohomology with respect to the U^k decomposition is just

$$GH^k_{\overline{\partial}} = \oplus_{p-q=k} H^q(M, \Omega^p(M)).$$

Finally, $d^{\mathcal{J}} = -i(\partial - \overline{\partial})$ is the standard d^c operator in the complex manifold.

2.2. The symplectic decomposition

In the symplectic case, the description of the spaces U^k is not as straightforward. However, we should notice that, if we let Λ be the interior product with the bivector $-\omega^{-1}$, then

$$d^{\mathcal{J}} = [d, \mathcal{J}] = d(\omega - \Lambda) - (\omega - \Lambda)d = \Lambda d - d\Lambda \tag{2.3}$$

is the operator introduced by Brylinski [1] and studied by Mathieu [10], Yan [12] and Merkulov [11], amongst others. One particular fact which will be useful later is that $d^{\mathcal{J}}$ commutes with Λ (see [12]).

Before we can state the precise form of the spaces U^k we need a lemma.

Lemma 2.2. *Let (V, ω) be a symplectic vector space. For any vector $X \in V \otimes \mathbb{C}$ and complex k-form α the following identities hold:*

$$\Lambda((i_X\omega) \wedge \alpha) = i_X\alpha + (i_X\omega) \wedge \Lambda\alpha;$$
$$2i\mathrm{e}^{\frac{\Lambda}{2i}}((i_X\omega) \wedge \alpha) = \mathrm{e}^{\frac{\Lambda}{2i}} i_X\alpha + 2i(i_X\omega)\mathrm{e}^{\frac{\Lambda}{2i}}\alpha.$$

Proof. We start with the first identity. It is enough to take Darboux coordinates so that ω is standard and check for $X = \partial_{x_i}$ and ∂_{y_i}. As both cases are similar we will do only the first.

Write $\alpha = \alpha_0 + dx_i\alpha_x + dy_i\alpha_y + dx_i \wedge dy_i\alpha_{xy}$. The left hand side is

$$\begin{aligned}\Lambda((i_X\omega) \wedge \alpha) &= \Lambda(dy_i \wedge \alpha) = \Lambda(dy_i\alpha_0 + dy_i \wedge dx_i\alpha_x) \\ &= dy_i\Lambda\alpha_0 + \alpha_x + dy_i \wedge dx_i\Lambda\alpha_x.\end{aligned}$$

And the right hand side is

$$\begin{aligned} i_X\alpha + (i_X\omega)\wedge\Lambda\alpha &= \alpha_x + dy_i\alpha_{xy} + dy_i(\Lambda\alpha_0 + dx_i\Lambda\alpha_x + dy_i\Lambda\alpha_y - \alpha_{xy} + dx_i\wedge dy_i\Lambda\alpha_{xy}) \\ &= \alpha_x + dy_i\alpha_{xy} + dy_i\Lambda\alpha_0 + dy_i\wedge dx_i\Lambda\alpha_x - dy_i\alpha_{xy}. \end{aligned}$$

So, the first identity follows.

By induction, from the first identity, we get that

$$\Lambda^k(i_X\omega\wedge\alpha) = k\Lambda^{k-1}i_X\alpha + (i_X\omega)\wedge\Lambda^k\alpha.$$

Therefore, by expanding the exponential in Taylor series, we obtain the second identity. □

Theorem 2.2. *The decomposition of* $\wedge^\bullet V \otimes \mathbb{C}$ *for a symplectic vector space* (V,ω) *is given by*

$$U^{n-k} = \{\mathrm{e}^{i\omega}(\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) \mid \alpha\in\wedge^k V\otimes\mathbb{C}\}.$$

Hence, the natural isomorphism

$$\varphi:\wedge^\bullet V\otimes\mathbb{C}\to\wedge^\bullet V\otimes\mathbb{C} \qquad \varphi(\alpha) = \mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\alpha,$$

is such that $\varphi:\wedge^k V\otimes\mathbb{C}\cong U^{n-k}$.

Proof. This is done by induction. For α a 0-form the expression above agrees with U^n. If U^{n-k} is as described above, then $U^{n-k-1} = \overline{L}\cdot U^{n-k}$, so its elements are linear combinations of terms of the form, with $\alpha\in\wedge^k V\otimes\mathbb{C}$,

$$\begin{aligned} (X + i i_X\omega)\mathrm{e}^{i\omega}(\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) &= \mathrm{e}^{i\omega}(i_X\mathrm{e}^{\frac{\Lambda}{2i}}\alpha + 2i(i_X\omega)\wedge\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) \\ &= 2i\mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}((i_X\omega)\wedge\alpha), \end{aligned}$$

where the second equality follows from the previous lemma. Now, since α can be chosen to be any k-form and ω is nondegenerate, the space generated by the forms above is $\{\mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\beta \mid \beta\in\wedge^{k+1}V\otimes\mathbb{C}\}$, and the theorem is proved. □

Now we move on to the operators ∂ and $\overline{\partial}$.

Theorem 2.3. *For any form* α,

$$d(\mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) = \mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\left(d\alpha - \frac{1}{2i}d^{\mathcal{J}}\alpha\right).$$

Therefore

$$\begin{aligned} \partial(\mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) &= -\mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\frac{1}{2i}d^{\mathcal{J}}\alpha \\ \overline{\partial}(\mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) &= \mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}d\alpha. \end{aligned}$$

Hence, the natural isomorphism φ *of Theorem 2.2 is such that*

$$\varphi(d\alpha) = \overline{\partial}\varphi(\alpha) \qquad \text{and} \qquad \varphi(d^{\mathcal{J}}\alpha) = -2i\partial\varphi(\alpha).$$

Proof. From Eq. (2.3), $d\Lambda = \Lambda d - d^{\mathcal{J}}$. Then, by induction, and using that $d^{\mathcal{J}}$ and Λ commute, $d\Lambda^k = \Lambda^k d - k\Lambda^{k-1}d^{\mathcal{J}}$. Therefore,

$$\begin{aligned} d(\mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) &= \mathrm{e}^{i\omega}d(\mathrm{e}^{\frac{\Lambda}{2i}}\alpha) = \mathrm{e}^{i\omega}\sum d\left(\frac{\Lambda^k}{(2i)^k k!}\alpha\right) \\ &= \mathrm{e}^{i\omega}\sum\left(\frac{\Lambda^k}{(2i)^k k!}d\alpha - \frac{1}{2i}\frac{\Lambda^{k-1}}{(2i)^{k-1}(k-1)!}d^{\mathcal{J}}\alpha\right) \\ &= \mathrm{e}^{i\omega}\mathrm{e}^{\frac{\Lambda}{2i}}\left(d\alpha - \frac{1}{2i}d^{\mathcal{J}}\alpha\right). \end{aligned}$$

The rest of the theorem follows from the fact that ∂ and $\overline{\partial}$ are the projections of d onto $\mathcal{U}^{k+1}$ and $\mathcal{U}^{k-1}$ respectively. □

Corollary 1. *As graded vector spaces, the $\overline{\partial}$-cohomology of a symplectic manifold M^{2n} is isomorphic to the de Rham cohomology*

$$GH^k_{\overline{\partial}}(M) = H^{n-k}(M;\mathbb{C}).$$

2.3. B-field transforms

According to Example 2.3, a generalized complex structure on a manifold can be deformed by a closed 2-form B. The canonical bundle of the deformed structure relates to the canonical bundle of the initial structure by

$$U^n_B = \mathrm{e}^B \cdot U^n$$

and the $-i$-eigenspace, $\overline{L_B}$ of the deformed structure relates to $\overline{L}$ via

$$\overline{L^B} = \{X + \xi - B(X,\cdot) : X + \xi \in \overline{L}\}.$$

From these, it is clear that $U^k_B = \mathrm{e}^B U^k$. Since B is closed, if $\alpha \in \mathcal{U}^k$, then

$$d(\mathrm{e}^B\alpha) = \mathrm{e}^B d\alpha = \mathrm{e}^B(\partial\alpha + \overline{\partial}\alpha) = \mathrm{e}^B\partial\alpha + \mathrm{e}^B\overline{\partial}\alpha,$$

where $\mathrm{e}^B\partial\alpha \in \mathcal{U}^{k+1}_B$ and $\mathrm{e}^B\overline{\partial}\alpha \in \mathcal{U}^{k-1}_B$. Therefore

$$\partial_B = \mathrm{e}^B\partial\mathrm{e}^{-B} \qquad \text{and} \qquad \overline{\partial}_B = \mathrm{e}^B\overline{\partial}\mathrm{e}^{-B}.$$

Using Gualtieri's version of Darboux's Theorem, these three cases can be used to describe the bundles U^k and the differentials ∂, $\overline{\partial}$ and $d^{\mathcal{J}}$ around regular points.

3. Generalized metric and Serre duality

Given a vector space V^n, a *metric in $V \oplus V^*$ compatible with the natural pairing*, which we will also call a *generalized metric*, is a self adjoint orthogonal transformation $\mathcal{G} \in \mathrm{End}(V \oplus V^*)$ such that

$$\langle \mathcal{G}v, v\rangle > 0 \qquad \text{if } v \in V \oplus V^* \setminus \{0\}.$$

As $\mathcal{G}$ is self adjoint, $\mathcal{G}^t = \mathcal{G}$, and orthogonality implies that $\mathcal{G}^t = \mathcal{G}^{-1}$, therefore $\mathcal{G}^2 = \mathrm{Id}$ and $\mathcal{G}$ splits $V \oplus V^*$ into its ± 1-eigenspaces, $C_\pm$. Conversely, giving two n-dimensional subspaces $C_\pm$ of $V \oplus V^*$ orthogonal with respect to the natural pairing such that the natural pairing is definite in $C_\pm$ furnishes $V \oplus V^*$ with a metric $\mathcal{G}$: just define $\mathcal{G}$ by letting $C_\pm$ be its ± 1-eigenspaces.

Finally, any n-dimensional subspace where the natural pairing is definite is a graph over V. This means that there are a symmetric form g and a skew symmetric form b for which

$$C_+ = \{X + (b+g)(X) : X \in V\}.$$

As the pairing is positive definite in C_+, g is a metric in V, and in order for C_- to be orthogonal to C_+ it must be the graph of $b - g$. Conversely, giving a metric g in V and a 2-form b determines the pair $C_\pm$ and hence the generalized metric $\mathcal{G}$.

Definition. Fix an orientation for C_+ and let $e_1, \ldots, e_n$ be an oriented orthonormal basis for this space. Denoting by τ the product $e_1 \cdots e_n \in \mathrm{Clif}(T \oplus T^*)$, the *generalized Hodge* $\star$ is defined by $\star\alpha = (-1)^{|\alpha|(n-1)}\tau \cdot \alpha$.

If we denote by $\star_g$ the usual Hodge star associated to the metric g, the Mukai pairing gives the following relation, if $b = 0$:

$$(\alpha, \star\beta) = \alpha \wedge \star_g \beta.$$

In the presence of a b-field, if we let $\alpha = \mathrm{e}^{-b}\tilde{\alpha}$ and $\beta = \mathrm{e}^{-b}\tilde{\beta}$, then the relation becomes

$$(\alpha, \star\beta) = \tilde{\alpha} \wedge \star_g \tilde{\beta}.$$

Hence $(\alpha, \star\alpha)$ is a nonvanishing volume from whenever $\alpha \neq 0$.

Definition. A metric $\mathcal{G} \in \mathrm{End}(V \oplus V^*)$ is *compatible* with a generalized complex structure $\mathcal{J}$ on V if $\mathcal{J}\mathcal{G} = \mathcal{G}\mathcal{J}$.

Similarly to the complex case, one can always find metrics compatible with a given generalized complex structure.

Lemma 3.1. *In a generalized complex manifold with compatible metric $(M, \mathcal{J}, \mathcal{G})$, the star operator preserves the U^k decomposition.*

Proof. Let $\mathcal{J}_2 = \mathcal{G}\mathcal{J}$. Then, as $\mathcal{G}$ and $\mathcal{J}$ commute, $\mathcal{J}_2$ is a generalized almost complex structure. If we denote by L_2 its $+i$-eigenspace, then one can easily check that

$$C_+ \otimes \mathbb{C} = L \cap L_2 \oplus \overline{L} \cap \overline{L_2}.$$

Therefore $\wedge^{2n} C_+ \otimes \mathbb{C} \cong \wedge^n(L \cap L_2) \otimes \wedge^n(\overline{L} \cap \overline{L_2})$. Since acting with an element of L in a form increases the U^k degree by 1 and acting with an element of $\overline{L}$ decreases by one, a volume element of C_+ preserves the U^k decomposition. □

For a generalized complex structure with compatible metric the operator $\bar{\star}$ defined by $\bar{\star}\alpha = \star\overline{\alpha}$ is also important, as it furnishes a definite, hermitian, bilinear functional,

$$h(\alpha, \beta) = \int_M (\alpha, \bar{\star}\beta), \qquad \alpha, \beta \in \Omega^{\bullet}_{\mathbb{C}}(M).$$

Lemma 3.2. *Let $(M, \mathcal{J}, \mathcal{G})$ be a generalized complex manifold with compatible metric. Then the h-adjoint of $\overline{\partial}$ is given by $\overline{\partial}^* = -\bar{\star}\overline{\partial}\bar{\star}^{-1}$.*

Proof. We start observing that $(d\alpha, \beta) + (\alpha, d\beta) = (d(\sigma(\alpha) \wedge \beta))_\top$ is an exact form. Now, let $\alpha \in \mathcal{U}^{k+1}$ and $\beta \in \mathcal{U}^{-k}$, then

$$(d(\sigma(\alpha) \wedge \beta))_\top = (d\alpha, \beta) + (\alpha, d\beta) = (\partial\alpha, \beta) + (\overline{\partial}\alpha, \beta) + (\alpha, \partial\beta) + (\alpha, \overline{\partial}\beta), \tag{3.1}$$

and according to Lemma 2.1, the terms $(\partial\alpha, \beta)$ and $(\alpha, \partial\beta)$ vanish. Therefore

$$\begin{aligned} h(\overline{\partial}\alpha, \beta) &= \int_M (\overline{\partial}\alpha, \star\overline{\beta}) = -\int_M (\alpha, \overline{\partial} \star \overline{\beta}) \\ &= -\int_M (\alpha, \bar{\star}\bar{\star}^{-1}\overline{\partial} \star \overline{\beta}) \\ &= h(\alpha, -\bar{\star}^{-1}\overline{\partial}\bar{\star}\beta). \quad \square \end{aligned}$$

Now, the Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is an elliptic operator and in a compact generalized complex manifold every $\overline{\partial}$-cohomology class has a unique harmonic representative, which is a $\overline{\partial}$ and $\overline{\partial}^*$-closed form. Also, from the expression above for $\overline{\partial}^*$, we see that $\bar{\star}$ maps harmonic forms to harmonic forms.

Theorem 3.1 (*Serre Duality*). *In a compact generalized complex manifold $(M^{2n}, \mathcal{J})$, the Mukai pairing gives rise to a pairing in cohomology $GH^k_{\overline{\partial}} \times GH^l_{\overline{\partial}} \to H^{2n}(M)$ which vanishes if $k \neq -l$ and is nondegenerate if $k = -l$.*

Proof. Given cohomology classes $a \in GH^k(M)$ and $b \in GH^l(M)$, choose representative $\alpha \in \mathcal{U}^k$ and $\beta \in \mathcal{U}^l$. According to Lemma 2.1, (α, β) vanishes if $k \neq -l$, therefore proving the first claim.

If $k = -l$ and $b = 0$, so that $\beta = \overline{\partial}\gamma$ is a $\overline{\partial}$-exact form, then, according to (3.1),

$$[(\alpha, \overline{\partial}\gamma)] = [\overline{\partial}\alpha, \gamma] = 0,$$

showing that the pairing is well defined.

Finally, if we let α be the harmonic representative of the class a, then $\bar{\star}\alpha$ is $\overline{\partial}$ closed form in U^{-k} which pairs nontrivially with α, showing nondegeneracy. □

4. The $dd^{\mathcal{J}}$-lemma and the cohomology decomposition

One property that helps to relate Dolbeault cohomology with ordinary cohomology in a complex manifold is the $\partial\overline{\partial}$-lemma. In this section we use the generalizations of the operators ∂ and $\overline{\partial}$ to define the analogue of the $\partial\overline{\partial}$-lemma for generalized complex manifolds and study cohomological implications of this lemma.

Definition. A generalized complex manifold *satisfies the $dd^{\mathcal{J}}$-lemma* if

$$\operatorname{Im} d \cap \ker d^{\mathcal{J}} = \operatorname{Im} d^{\mathcal{J}} \cap \ker d = \operatorname{Im} dd^{\mathcal{J}}.$$

Remark. We could equivalently have said that M satisfies the $\partial\overline{\partial}$-lemma if

$$\operatorname{Im} \partial \cap \ker \overline{\partial} = \operatorname{Im} \overline{\partial} \cap \ker \partial = \operatorname{Im} \partial\overline{\partial}.$$

It is easy to see that these properties are equivalent.

Lemma 4.1. *If* $\operatorname{Im} d^{\mathcal{J}} \cap \ker d = \operatorname{Im} dd^{\mathcal{J}}$ *then the $dd^{\mathcal{J}}$-lemma holds.*

Proof. As $d^{\mathcal{J}} = \mathcal{J}^{-1} d \mathcal{J}$, if $\alpha \in \operatorname{Im} d \cap \ker d^{\mathcal{J}}$, then $\mathcal{J}\alpha \in \operatorname{Im} d^{\mathcal{J}} \cap \ker d$ and hence $\mathcal{J}\alpha = dd^{\mathcal{J}}\beta$, for some β. Therefore

$$\alpha = -\mathcal{J} dd^{\mathcal{J}}\beta = \mathcal{J} d \mathcal{J}^{-1} d \mathcal{J}\beta = d^{\mathcal{J}} d(\mathcal{J}\beta),$$

and hence $\alpha \in \operatorname{Im} dd^{\mathcal{J}}$. □

Theorem 4.1. *The following properties are equivalent for a generalized complex manifold $(M, \mathcal{J})$:*

1. *M satisfies the $dd^{\mathcal{J}}$-lemma;*
2. *The inclusion of the complex of $d^{\mathcal{J}}$-closed forms $\Omega_{d^{\mathcal{J}}}$ into the complex of differential forms Ω induces an isomorphism in cohomology:*

$$(\Omega_{d^{\mathcal{J}}}, d) \overset{i}{\hookrightarrow} (\Omega, d), \qquad H^{\bullet}(\Omega_{d^{\mathcal{J}}}) \overset{i^*}{\cong} H^{\bullet}(\Omega).$$

Proof. We start with the implication (1) ⇒ (2).

(i) $i^* : H^{\bullet}(\Omega_{d^{\mathcal{J}}}) \to H^{\bullet}(\Omega)$ is injective:
If $i^*\alpha$ is exact, then α is $d^{\mathcal{J}}$-closed and exact, hence by the $dd^{\mathcal{J}}$-lemma $\alpha = dd^{\mathcal{J}}\beta$, so α is the derivative of a $d^{\mathcal{J}}$-closed form and hence its cohomology class in $\Omega_{d^{\mathcal{J}}}$ is also zero.

(ii) $i^* : H^{\bullet}(\Omega_{d^{\mathcal{J}}}) \to H^{\bullet}(\Omega)$ is surjective:
Let α be a closed form and set $\beta = d^{\mathcal{J}}\alpha$. Then $d\beta = dd^{\mathcal{J}}\alpha = -d^{\mathcal{J}} d\alpha = 0$, so β satisfies the conditions of the $dd^{\mathcal{J}}$-lemma, hence $\beta = d^{\mathcal{J}} d\gamma$. Let $\tilde{\alpha} = \alpha - d\gamma$, then $d^{\mathcal{J}}\tilde{\alpha} = d^{\mathcal{J}}\alpha - d^{\mathcal{J}} d\gamma = \beta - \beta = 0$, so $[\alpha] \in \operatorname{Im}(i^*)$.

For the converse, we will prove that $\operatorname{Im} d^{\mathcal{J}} \cap \ker d = \operatorname{Im} dd^{\mathcal{J}}$ which, according to Lemma 4.1, is equivalent to the $dd^{\mathcal{J}}$-lemma. Let $d^{\mathcal{J}}\alpha \in \operatorname{Im} d^{\mathcal{J}} \cap \ker d$. We want to prove that $d^{\mathcal{J}}\alpha = dd^{\mathcal{J}}\beta$, for some β. Since $d^{\mathcal{J}} d\alpha = 0$, $d\alpha \in \mathcal{E}_{d^{\mathcal{J}}}$ is a closed form in $\Omega_{d^{\mathcal{J}}}$ and represents the trivial cohomology class in Ω, hence it also represents the trivial cohomology class in $\Omega_{d^{\mathcal{J}}}$, i.e., there is $\gamma_1 \in \Omega_{d^{\mathcal{J}}}$ such that $d\alpha = d\gamma_1$. Therefore $d(\alpha - \gamma_1) = 0$ and the cohomology class $[\alpha - \gamma_1]$ has a $d^{\mathcal{J}}$-closed representative γ_2:

$$\alpha - \gamma_1 = \gamma_2 + d\beta.$$

Applying $d^{\mathcal{J}}$, we get $d^{\mathcal{J}}\alpha = d^{\mathcal{J}} d\beta$, as we wanted. □

Corollary 2. *If the $dd^{\mathcal{J}}$-lemma holds, the splitting $\Omega^{\bullet}(M) = \oplus \mathcal{U}^k$ gives rise to a splitting of cohomology, i.e., any cohomology class $a \in H^{\bullet}(M, C)$ can be represented by a form $\alpha = \sum \alpha_k$, with $\alpha_k \in \mathcal{U}^k$ such that $d\alpha_k = 0$. If $a = 0$ is the trivial cohomology class, then for any such α each α_k is exact.*

Proof. Let a be a cohomology class. From the previous theorem, there is a representative α for it which is d- and $d^{\mathcal{J}}$-closed. Since $d = \partial + \overline{\partial}$ and $d^{\mathcal{J}} = -i(\partial - \overline{\partial})$, we conclude that α is both ∂ and $\overline{\partial}$ closed, and so must be each of its components α_k relative to the splitting. Hence we obtain a splitting for the cohomology class $a = \sum [\alpha_k]$. If a was the trivial cohomology class, any such α would be d-exact and $d^{\mathcal{J}}$-closed, hence, equal to $dd^{\mathcal{J}}\beta$, for some β. So the decomposition of α would be $\alpha_k = dd^{\mathcal{J}}\beta_k$, showing that each of the summands is exact. □

Definition. If $(M, \mathcal{J})$ is a generalized complex manifold satisfying the $dd^{\mathcal{J}}$-lemma, we define the *generalized cohomology of* M, $GH^k(M)$, as the cohomology classes in $H^\bullet(M)$ that can be represented by elements of $\mathcal{U}^k$.

In this case, Serre duality furnishes the following.

Corollary 3 (*Poincaré Duality*). *On a compact generalized complex manifold M satisfying the $dd^{\mathcal{J}}$-lemma, the Mukai pairing vanishes in $GH^k(M) \times GH^l(M)$ unless $k = -l$, in which case it is nondegenerate.*

5. The canonical spectral sequence

The decomposition of $d = \partial + \overline{\partial}$ gives rise to a spectral sequence similar to the Frölicher spectral sequence of a complex manifold. The object of this section is the study of this spectral sequence. The only subtlety is that while in the complex case there is a natural bigrading for the complex of differential forms, we have found only one grading for forms on a generalized complex manifold, namely, the one given by the U^k. A way to remedy this is to mimic Brylinski [1] and Goodwillie [5]: introduce a formal element β of degree 2 and consider the complex:

$$\mathcal{A} = \Omega^\bullet(M) \otimes \odot\mathrm{span}\{\beta, \beta^{-1}\},$$

and to change the differential to:

$$d^\beta(a\beta^k) = \partial a\beta^k + \overline{\partial} a\beta^{k+1}.$$

The complex $\mathcal{A}$, which we call the *canonical complex*, has a bigrading given by $\mathcal{A}^{p,q} = \mathcal{U}^{p-q}\beta^q$, and the differential d^β decomposes as ∂^β and $\overline{\partial}^\beta$, where $\partial^\beta : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\overline{\partial}^\beta : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$. The complex of differential forms sits inside $\mathcal{A}$ as the β-periodic elements:

$$\tau : \Omega \to \mathcal{A}; \qquad \tau(\alpha) = \sum_{k\in\mathbb{Z}} \alpha\beta^k.$$

And this is a map of differential algebras which preserves the decompositions of d and d^β:

$$\tau(\partial\alpha) = \partial^\beta\tau(\alpha) \qquad \text{and} \qquad \tau(\overline{\partial}\alpha) = \overline{\partial}^\beta\tau(\alpha).$$

One can easily check that the $\partial\overline{\partial}$-lemma holds for $\Omega^\bullet$ if and only if the corresponding lemma holds for $\mathcal{A}$.

Also, the bigrading gives a filtration of $\mathcal{A}$:

$$F^p\mathcal{A} = \sum_{p'\geq p} \mathcal{A}^{p',q};$$
$$F^p\mathcal{A}^m = \sum_{p'\geq p} \mathcal{A}^{p',m-p'},$$

which is preserved by d^β, i.e., $d^\beta : F^p\mathcal{A} \to F^p\mathcal{A}$. For each m, $F^p\mathcal{A}^m = \{0\}$ for $2p \geq n + m$ and $F^p\mathcal{A}^m = \mathcal{A}^m$ for $2p \leq m - n$, where $2n$ is the dimension of the manifold. This means that the filtration is *bounded* and hence the induced spectral sequence, which we call the *canonical spectral sequence*, converges to the cohomology of the operator d^β. This spectral sequence is periodic in the sense that $E_r^{p,q} \cong E_r^{p-q,0}$.

Then the first term $E_1^{p,q} \cong GH_{\overline{\partial}}^{p-q}$ is just the $\overline{\partial}$-cohomology of the manifold, which is finite dimensional, since $\overline{\partial}$ defines an elliptic differential complex. The second term is the cohomology induced by ∂ in $H_{\overline{\partial}}$, and the sequence goes on. However, if the $\partial\overline{\partial}$-lemma holds this sequence degenerates at E_1. Conversely, Deligne's 'theorem' ([3], Proposition 5.17 and Remark 5.21) tells us that the degeneracy at E_1 together with the decomposition of cohomology imply the $dd^{\mathcal{J}}$-lemma:

Theorem 5.1. *If the canonical spectral sequence degenerates at E_1 and the decomposition of forms into subbundles U^k induces a decomposition in cohomology, then the $dd^{\mathcal{J}}$-lemma holds.*

Remark. For a generalized complex structure induced by a complex structure the canonical spectral sequence is just the Frölicher spectral sequence repeated over and over. In particular, the degeneracy of the canonical spectral sequence at E_r is equivalent to the degeneracy of the Frölicher spectral sequence at the same stage.

It is possible that the canonical spectral sequence degenerates at E_1 even though the $dd^{\mathcal{J}}$-lemma does not hold. One example of such a phenomenon is given by a result by Kodaira [9] stating that the Frölicher—and hence the canonical—spectral sequence always degenerates at E_1 for complex surfaces, although not all of those satisfy the dd^c-lemma.

In the symplectic case, the canonical spectral sequence always degenerates at E_1, as we show next (this is also a consequence of a more complicated argument of Brylinski [1]).

Theorem 5.2. *In a symplectic manifold, the canonical spectral sequence degenerates at E_1.*

Proof. The term E_1 of the canonical spectral sequence is just the $\overline{\partial}$-cohomology which is isomorphic to the d-cohomology, according to Theorem 2.3, and which is the final stage of the spectral sequence. Therefore $E_1 \cong E_\infty$ and the sequence converges after the first step. □

Remark. According to Theorem 2.3, the canonical spectral sequence for a symplectic manifold, obtained from the decomposition $d = \partial + \overline{\partial}$ is isomorphic to the sequence obtained from d and $d^{\mathcal{J}}$, using degree of forms for grading. The latter spectral sequence was the one studied by Brylinki [1].

Finally, following Frölicher [4], we compute the Euler characteristic from the $\overline{\partial}$-cohomology.

Proposition 5.1. *If M^{2n} admits a generalized complex structure, then the Euler characteristic of M is given by*

$$\chi(M) = \pm \sum (-1)^k \dim GH^k_{\overline{\partial}}(M),$$

where the sign is $+$ if the elements in U^0 are even forms and $-$ otherwise.

Proof. Given the periodic condition, $E_r^{p,q} \cong E_r^{p-q,0}$, this spectral sequence is equivalent to long exact sequences:

$$\begin{aligned} &\cdots \to \mathcal{U}^{k-1} \overset{d_1=\partial}{\to} \mathcal{U}^k \overset{d_1=\partial}{\to} \mathcal{U}^{k+1} \to \cdots; \\ &\cdots \to H_\partial^{k-3} \overset{d_2}{\to} H_\partial^k \overset{d_2}{\to} H_\partial^{k+3} \to \cdots; \\ &\cdots \to H_{d_{r-1}}^{k-2r+1} \overset{d_r}{\to} H_{d_{r-1}}^k \overset{d_r}{\to} H_{d_{r-1}}^{k+2r-1} \to \cdots. \end{aligned}$$

As d_r maps ev/od to od/ev, the Euler characteristic is preserved and hence can be computed from the first sequence where the spaces involved are finite dimensional. □

6. Submanifolds

In this section we prove that generalized complex submanifolds are represented by elements of $GH^0(M)$, whenever the cohomology of M splits.

Definition. A manifold with 2-form (N, F) is a *generalized complex submanifold* of a generalized complex manifold $(M, \mathcal{J})$ if $dF = 0$ and the tangent space

$$\tau_F = \{X + \xi \in TM \oplus T^*M : X \in TN \text{ and } F(X, \cdot) = \xi|_{TN}\}$$

is invariant with respect to $\mathcal{J}$.

Particular examples of generalized complex submanifolds are complex submanifolds, in the case of a generalized complex structure induced by a complex structure, and Lagrangian submanifolds of symplectic manifolds.

Lemma 6.1. *In a generalized complex vector space $(V, \mathcal{J})$, if a complex valued form ρ annihilates a maximal isotropic of $W \subset V \oplus V^*$ invariant under $\mathcal{J}$ then $\rho \in U^0$.*

Remark. The point is that W is *real*.

Proof. As the space W annihilated by ρ is a real maximal isotropic, ρ is just the complex multiple of a real form, and so is any other form annihilating W. Therefore we can assume, without loss of generality, that ρ is real. For $v \in W$, let $\mathcal{J}$ act via the Lie algebra action, so

$$0 = \mathcal{J}(v \cdot \rho) = \mathcal{J}v \cdot \rho + v \cdot \mathcal{J}\rho = v \cdot \mathcal{J}\rho, \quad \forall v \in W.$$

Therefore $\mathcal{J}\rho$ is a multiple of ρ, say $\mathcal{J}\rho = ik\rho$ and $\rho \in U^k$. But as ρ is real, $\rho = \overline{\rho} \in U^{-k}$, hence $k = 0$. □

Theorem 6.1. *Let M be a compact generalized complex manifold for which the cohomology decomposes in generalized cohomology and let (N, F) be a generalized submanifold. Then $[\mathrm{e}^{-F} PD(N)]$ is a class in $GH^0(M)$.*

Remark. By the definition of generalized complex submanifold, the form F is defined only on N, but by the Thom isomorphism theorem and the identification of the Thom class of a tubular neighbourhood of N with the Poincaré dual of N we see that $[\mathrm{e}^{F} PD(N)]$ is a well defined cohomology class on M.

Proof. Let the submanifold be given locally by the vanishing of coordinates $x_1 = \cdots = x_k = 0$, so that τ_F is annihilated by $\mathrm{e}^{-F} dx_1 \wedge \cdots \wedge dx_k$ and let $\alpha \in \mathcal{U}^k$, $k \neq 0$. Then, according to Lemmas 2.1 and 6.1, at a point in the submanifold,

$$0 = (\mathrm{e}^{-F} dx_1 \wedge \cdots \wedge dx_k, \alpha) = (dx_1 \wedge \cdots \wedge dx_k, \mathrm{e}^{F}\alpha).$$

As $dx_1 \wedge \cdots \wedge dx_k$ is of degree k, this means that the wedge product of the degree $2n - k$ component of $\mathrm{e}^F\alpha$ with a volume form for the conormal bundle of N vanishes. This is the same as saying that the restriction of the degree $2n - k$ part of $\mathrm{e}^F\alpha$ to N vanishes and therefore

$$\int_N \mathrm{e}^F \alpha = 0.$$

Letting $PD(N)$ be the Poincaré dual of N and $a \in GH^k(M)$, $k \neq 0$, we have

$$\int_M (\mathrm{e}^{-F} PD(N), \alpha) = \int_M (PD(N), \mathrm{e}^F \alpha) = \int_N \mathrm{e}^F \alpha = 0.$$

Showing that $\mathrm{e}^{-F} PD(N)$ pairs trivially with any cohomology class in $GH^k(M)$ and hence lies in $GH^0(M)$. □

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